MTH 304: Metric Spaces and Topology

Practice Assignment V: Separation axioms

1. Reading assignment: Read through the following proofs:
   (a) $\mathbb{R}_\ell^2$ is not normal (see Example 3, page 196 in Munkres).
   (b) $\mathbb{R}^J$ is not normal (see Problem 9, page 204 in Munkres).

2. Let $f, g : X \to Y$ be continuous maps. Then show that set $\{x \in X \mid f(x) = g(x)\}$ is closed in X.

3. Let $p : X \to Y$ be a closed continuous and surjective map. Show that if X is normal, then so is Y.

4. A closed continuous and surjective map $p : X \to Y$ is said to be *perfect* if for each $y \in Y$, $p^{-1}(\{y\})$ is a compact subset of X. Let $p : X \to Y$ be a perfect map.
   (a) Show that if X is regular, then so is Y.
   (b) If X is locally compact, then so is Y.
   (c) If X is second countable, then so is Y.

5. Let G be a compact topological group, and let X be a topological space such that $G \curvearrowright X$ (i.e G acts on X). Then show that:
   (a) If X is normal, then so is X/G.
   (b) If X is locally compact, then so is X/G.
   (c) If X is second countable, then so is X/G.

6. Show that if $\prod_{\alpha \in J} X_\alpha$ is normal, then so is X_α, for each $\alpha \in J$.

7. Show that a locally compact Hausdorff space is regular.

8. Show that a regular Lindelöf space is normal.

9. A space X is *completely normal* if every subspace of X is normal. Show that a space X is completely normal if, and only if for every pair $\{A, B\}$ of subsets of X such that
$$\bar{A} \cap B = A \cap \bar{B} = \emptyset,$$
can be separated by open sets.

10. A T_1 space X is said to be *completely regular* if for each point $x \in X$, and each closed set $A \subset X \setminus \{x\}$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.
    (a) Show that subspaces and products of completely regular spaces are completely regular. (See Theorem 33.2, Munkres).
    (b) Show that a locally compact Hausdorff space is completely regular.
    (c) $\mathbb{R}^J$ with the box topology is completely regular.

(d) Every topological group is completely regular. (See Problem 10, page 211 in Munkres.)

11. Let X be a compact Hausdorff space. Show that X is metrizable if, and only if X is second countable.

12. A space X is said to be *locally metrizable* if each point $x \in X$ has a neighborhood that is metrizable in the subspace topology.

    (a) Show that a compact Hausdorff space is metrizable if, and only if its locally metrizable.

    (b) Show that a regular Lindelöf space is metrizable if, and only if its locally metrizable.

13. Show that in a metrizable space X, the following are equivalent.

    (a) X is bounded under the topology $\mathcal{T}_d$ induced by any metric d.

    (b) Every continuous function $\phi : X \to \mathbb{R}$ is bounded.

    (c) X is limit point compact.

14. let X be a topological space, let $A \subset X$. Then a *retraction of* X *into* A is a continuous map $f : X \to A$ such that $f|_A = i_A$. A subspace A of a topological X is called a *retract* of X if there exists a retraction $r : X \to A$.

    (a) Show that the retract of a Hausdorff space is closed.

    (b) Show that no two-point subset of $\mathbb{R}^2$ can be retract of $\mathbb{R}^2$.

    (c) Show that S^1 is a retract of $\mathbb{R}^2 \setminus \{0\}$.